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Painlevé analysis of the two-dimensional Burgers equation

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Abstract. A Painlevé analysis of the two-dimensional Burgers equation is carried out and used to obtain a restricted Bäcklund transformation that maps a subclass of the solutions of the 1+2D Burgers equation onto a linear heat-like equation. Alternatively, the Bäcklund transformation can be expressed as a map onto the derivative of the one-dimensional Burgers equation in appropriate dependent and independent variables. The singularity analysis also yields a further class of solutions obtained by solving a Schwarzian differential equation.

1. Introduction

Painlevé analysis of partial differential equations (e.g. Weiss *et al* [1], Weiss [2,3], Newell *et al* [4] and references therein, Conte [5]) is a powerful tool for uncovering the integrability properties of nonlinear systems of differential equations. In the present paper, we carry out a Painlevé analysis of the two-dimensional Burgers equation

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0 \tag{1.1}$$

which describes weakly nonlinear two-dimensional shocks in dissipative media. The shocks described by equation (1.1) are weakly two-dimensional in the sense that the scale length of variation in the y direction is much larger than in the x direction. A general derivation of (1.1) has been given by Bartucelli *et al* [6]. Equation (1.1) is sometimes referred to as the Zabolotskaya–Khoklov equation in nonlinear acoustics (Zabolotskaya and Khoklov [7], Rudenko and Soluyan [8], Crighton [9] and Hunter [10]), with the u_{yy} term representing wave diffraction. Application of the 1+2D Burgers equation to weak shocks modified by the first-order Fermi acceleration of energetic particles in cosmic-ray astrophysics has been carried out by Zank and Webb [11]. Segur [12] has pointed out that equation (1.1) is of considerable interest as a 1+2D nonlinear wave equation.

The main result obtained from the Painlevé analysis is a 'restricted' Bäcklund transformation that maps a subclass of the solutions of equation (1.1) onto a linear heat-like equation. We show that equation (1.1) possesses solutions of the form

$$u = -2\frac{\vartheta_h}{\vartheta} + f'(t)y - [f(t)]^2 \tag{1.2}$$

where

$$h = x - f(t)y \tag{1.3}$$

and ϑ satisfies the linear heat-like equation

$$\vartheta_t - \vartheta_{hh} + [p(t)h + q(t)]\,\vartheta = 0. \tag{1.4}$$

The functions $f(t)$, $p(t)$ and $q(t)$ in the Bäcklund transformation (1.2)–(1.4) are arbitrary. It transpires that the 1+2D Burgers equation has the conditional Painlevé property. The singularity analysis also yields a further class of solutions distinct from the Bäcklund transformation solutions (1.2)–(1.4).

In section 2, the recurrence relations and resonances of the Painlevé expansion are established. In section 3, we obtain the Bäcklund transformation and other special solutions. Then, in section 4, we indicate the relation of the results of the conventional Painlevé analysis of section 3 to the invariant Painlevé expansion developed by Conte [5]. In brief, the Painlevé expansions used are of the form

$$u = \phi^{-p}\sum_{j=0}^{\infty} u_j\phi^j \equiv \chi^{-p}\sum_{j=0}^{\infty}\tilde{u}_j\chi^j \tag{1.5}$$

where

$$\chi = [\phi_x/\phi - \phi_{xx}/(2\phi_x)]^{-1} \tag{1.6}$$

and $\phi = 0$ defines the singularity manifold. The expansion in terms of ϕ is the conventional Painlevé expansion, whereas the equivalent expansion in terms of χ is Conte's invariant expansion. By construction the coefficients $\tilde{u}_j$ of the invariant expansion are invariant under the homographic (or Möbius) transformation

$$\phi' = \frac{a\phi + b}{c\phi + d} \qquad ad - bc \neq 0. \tag{1.7}$$

For the 1+2D Burgers equation, the $\tilde{u}_j$ depend on the homographic invariants

$$C = -\phi_t/\phi_x \qquad W = -\phi_y/\phi_x \qquad S = \{\phi; x\} \tag{1.8}$$

where

$$\{\phi; x\} = \frac{\partial}{\partial x}\left(\frac{\phi_{xx}}{\phi_x}\right) - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{1.9}$$

is the Schwarzian derivative (e.g. Hille [13]). Although the invariant expansion leads to the Bäcklund transformation (1.2)–(1.4) fairly quickly, it is less successful in suggesting the second class of solutions not covered by the Bäcklund transformation. Section 5 concludes the paper with a brief discussion.

2. Painlevé expansion

Following the approach of Weiss *et al* [1] and Weiss [2, 3], we use the singular manifold expansion

$$u = \phi^{-p}\sum_{j=0}^{\infty} u_j(x, y, t)\phi^j \tag{2.1}$$

in the Painlevé analysis of the 2D Burgers equation (1.1). Here $\phi(x,y,t)=0$ defines the singular manifold. The balancing of the powers of lowest order requires $p=1$.

By balancing terms of order ϕ^{j-4} in (1.1), we obtain the recurrence relations

$$\begin{aligned}(j-2)(j-3)\phi_x\phi_t u_{j-1} &+ (j-3)[u_{j-2,t}\phi_x + u_{j-2,x}\phi_t + u_{j-2}\phi_{tx}] + u_{j-3,tx}\\ &+ \sum_{k=0}^{j}\{(k-1)(j-k-1)\phi_x^2 u_{j-k}u_k + 2(j-k-2)\phi_x u_{k,x}u_{j-k-1}\\ &+ u_{k,x}u_{j-k-2,x} + (k-1)(k-2)\phi_x^2 u_k u_{j-k}\\ &+ (k-1)u_{j-k-1}[\phi_{xx}u_k + 2\phi_x u_{k,x}] + u_{j-k-2}u_{k,xx}\}\\ &- \{(j-1)(j-2)(j-3)\phi_x^3 u_j + (j-2)(j-3)[3\phi_x\phi_{xx}u_{j-1} + 3\phi_x^2 u_{j-1,x}]\\ &+ (j-3)[\phi_{3x}u_{j-2} + 3\phi_{xx}u_{j-2,x} + 3\phi_x u_{j-2,xx}] + u_{j-3,xxx}\Big\}\\ &+ (j-2)(j-3)\phi_y^2 u_{j-1} + (j-3)[\phi_{yy}u_{j-2} + 2\phi_y u_{j-2,y}] + u_{j-3,yy} = 0\end{aligned} \tag{2.2}$$

where coefficients u_s having negative subscripted indices are taken to be zero.

For $j=0$, equation (2.2) yields

$$u_0 = -2\phi_x. \tag{2.3}$$

On using (2.3) for u_0 in the recurrence relations (2.2), we find that resonances occur at $j=-1$, 2 and 3. The recurrence relations at orders $j=1,2,3$ may be expressed as

$$j=1: \qquad u_1 = \frac{\phi_{xx}}{\phi_x} - \left(\frac{\phi_y}{\phi_x}\right)^2 - \frac{\phi_t}{\phi_x} \tag{2.4}$$

$$j=2: \qquad \phi_y^2\phi_{xx} + \phi_x^2\phi_{yy} - 2\phi_x\phi_y\phi_{xy} = 0 \tag{2.5}$$

$$j=3: \qquad u_{0,tx} + (u_0u_1)_{xx} - u_{0,xxx} + u_{0,yy} = 0. \tag{2.6}$$

The resonance equation (2.6) at order $j=3$ is automatically satisfied for u_0, u_1 and ϕ satisfying equations (2.3)–(2.5). A consistent truncation of the Painlevé expansion (2.1) is obtained if we set $u_j=0$ for $j\geq 2$, in which case the $j=4$ balance equation reduces to

$$j=4: \qquad [u_{1,t} + u_1u_{1,x} - u_{1,xx}]_x + u_{1,yy} = 0 \tag{2.7}$$

so that u_1 also satisfies the 2D Burgers equation (1.1). The higher-order balance equations for $j\geq 5$ are satisfied identically if we set $u_j=0$ for $j\geq 2$. The truncated Painlevé expansion (2.1) then becomes

$$u = -2\frac{\phi_x}{\phi} + u_1. \tag{2.8}$$

Since the condition (2.5) at the $j=2$ resonance is not automatically satisfied, and imposes a constraint on ϕ, we observe that the 2D Burgers equation has the conditional Painlevé property (e.g. [2, 14]). We note that the recurrence relations (2.2) can be written in the form

$$(j-\beta_1)(j-\beta_2)(j-\beta_3)u_j = F_j\left(\phi_x,\phi_y,\phi_t;u_0,u_1,u_2,\ldots,u_{j-1}\right) \tag{2.9}$$

where $\beta_1 = -1$, $\beta_2 = 2$ and $\beta_3 = 3$. Resonances occur when $j = \beta_1$, β_2 or β_3. For an equation to have the Painlevé property, the resonant β_i have integer values and the F_j are identically zero at the resonances. The conditional Painlevé property arises if the resonances are integers, but the F_j at the resonances are not identically zero. In this case, the resonant F_j can be made to vanish either by including logarithms in the expansion or by constraining ϕ to satisfy $F_j = 0$ at the resonances. In the present analysis, F_2 (given by the left-hand side of equation (2.5)) is not identically zero at the $j = 2$ resonance, and so the 1+2D Burgers equation has the conditional Painlevé property. The 1+2D Burgers equation does not have the Painlevé property.

3. Solutions

We proceed to determine solutions of the 1+2D Burgers equation (1.1) by solving the recurrence relations (2.3)–(2.7). Since u_1 is given explicitly in terms of ϕ by equation (2.4), equation (2.7) can consequently be regarded as an equation for ϕ. Thus, ϕ must satisfy simultaneously the two equations (2.5) and (2.7). We first determine the general solution of equation (2.5) in implicit form giving x as a function of ϕ, y and t, i.e. $x = X(\phi, y, t)$ for some appropriate function X. Substitution of this solution in equation (2.7) then yields a single equation for $X(\phi, y, t)$. Solution of the equation for X leads to the Bäcklund transformation (1.2)–(1.4), and to a further special class of solutions for ϕ of the form $\phi = \Phi(A)$, where $A(x, y, t)$ is a specific function of x, y and t.

3.1. Solution of constraint equation (2.5)

The quasi-linear partial differential equation (2.5) for ϕ can be solved by noting that the equation is equivalent to the system of first-order equations

$$W = -\frac{\phi_y}{\phi_x} \qquad W_y + WW_x = 0. \tag{3.1}$$

The second of equations (3.1) can be solved by the method of characteristics (e.g. Sneddon [15]), and has the general solution

$$W = \Omega(x - Wy, t) \tag{3.2}$$

where Ω is an arbitrary differentiable function. A combination of equations (3.1) yields $\partial(W, \phi)/\partial(x, y) = 0$, so that $W = f(\phi, t)$ for some arbitrary function f. Hence, the general solution of equation (2.5) is

$$x = X(\phi, y, t) \equiv f(\phi, t)y + h(\phi, t) \tag{3.3}$$

where $h \equiv \Omega^{-1} \circ f$ (an alternative derivation of the solution (3.3) using a method due to Monge is given in Sneddon [15], ch. 13, section 11). Equation (2.5) is also discussed by Weiss [2] (his equation (3.21)). We note that W is a homographic invariant (see equations (1.8), and section 4), with functional form

$$W = f(\phi, t). \tag{3.4}$$

3.2. Reduction of equation (2.7) to an equation for $X(\phi, y, t)$

To proceed further with the analysis, we regard x as the dependent variable, with $x = X(\phi, y, t)$ having the functional form (3.3), which ensures that ϕ satisfies (2.5). Equation (2.4) then yields

$$u_1 = -\frac{X_{\phi\phi}}{X_\phi^2} - f^2 + X_t. \tag{3.5}$$

By regarding $x \equiv X$ as the dependent variable in the 2D Burgers equation (2.7) for u_1, we find

$$\begin{aligned} X_\phi^8 \left\{ \left[u_{1,t} + u_1 u_{1,x} - u_{1,xx}\right]_x + u_{1,yy} \right\} &= X_\phi^7 \left(X_{tt\phi} - 2f_\phi f_t - 4ff_{t\phi}\right) \\ &+ X_\phi^5 \left(-2X_{\phi\phi\phi t} + 10 f_\phi f_{\phi\phi} + 4ff_{3\phi}\right) \\ &+ X_\phi^4 \{6X_{\phi\phi}X_{\phi\phi t} + 2X_{3\phi}X_{\phi t} - 4ff_\phi X_{3\phi} \\ &- \left[12ff_{\phi\phi} + 10(f_\phi)^2\right] X_{\phi\phi}\} + X_\phi^3 \left(-6X_{\phi\phi}^2 X_{\phi t} + X_{5\phi} + 12ff_\phi X_{\phi\phi}^2\right) \\ &+ X_\phi^2 \left(-10X_{\phi\phi}X_{4\phi} - 6X_{3\phi}^2\right) + 48X_\phi X_{\phi\phi}^2 X_{3\phi} - 36X_{\phi\phi}^4 = 0. \end{aligned} \tag{3.6}$$

To analyze equation (3.6), it is necessary to consider the cases $f_\phi \neq 0$ and $f_\phi = 0$ separately. Solutions of equation (3.6) with $f_\phi \neq 0$, lead to the special solutions of the form $\phi = \Phi(A)$, whereas the solutions of equation (3.6) with $f_\phi = 0$ yield the Bäcklund transformation (1.2)–(1.4).

3.3. Case $f_\phi \neq 0$

Proposition 1. Assume that $f_\phi \neq 0$. Then the 1+2D Burgers equation has solutions of the form

$$u = \frac{-2Y'(A)}{Y(A)} \frac{1}{y + g(t)} + \tfrac{1}{2}g''(t)y + m'(t) + g'(t)A - \left(\tfrac{1}{2}g'(t) + k_0 + A\right)^2 \tag{3.7}$$

where

$$A = \frac{x - \left[\frac{1}{2}g'(t) + k_0\right] y - m(t)}{y + g(t)} \tag{3.8}$$

and $Y(A)$ is a solution of Airy's equation

$$Y''(A) + (k_1 A + k_2) Y(A) = 0. \tag{3.9}$$

In equations (3.7)–(3.9), $g(t)$ and $m(t)$ are arbitrary differentiable functions of t and k_0, k_1 and k_2 are arbitrary constants.

To obtain the solutions described in Proposition 1 above, we solve equation (3.6) for X. We first write

$$X_\phi = f_\phi y + h_\phi = f_\phi (y + g) \tag{3.10}$$

where

$$g = h_\phi / f_\phi. \tag{3.11}$$

We then note, for example, that

$$X_{\phi\phi} = \frac{f_{\phi\phi}}{f_\phi} X_\phi + h_{\phi\phi} - g f_{\phi\phi} \tag{3.12}$$

(and similarly for the other derivatives of X_ϕ in equation (3.6)). Equation (3.6) can then be expressed either as a polynomial equation of degree eight in y, or alternatively as a polynomial of degree eight in X_ϕ. Equating powers of X_ϕ to zero leads to a set of nine determining equations for f and g (see appendix 1). It transpires that the determining equations require either $g_\phi = 0$ or $f_\phi = 0$. Since we have assumed $f_\phi \neq 0$ from the outset, then $g_\phi = 0$. Solving the determining equations yields solutions for f, g and h of the form

$$g = g(t); \tag{3.13}$$

$$f = A(\phi) + \tfrac{1}{2} g'(t) + k_0 \tag{3.14}$$

$$h = g(t)A(\phi) + m(t) \tag{3.15}$$

where $g(t)$ and $m(t)$ are arbitrary functions of t; k_0 is an arbitrary constant. The function $A(\phi)$ satisfies the fifth-order differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\{A;\phi\} - 5\frac{A''(\phi)}{A'(\phi)}\frac{\mathrm{d}}{\mathrm{d}\phi}\{A;\phi\} + 5\left[\frac{A''(\phi)}{A'(\phi)}\right]^2 \{A;\phi\} - 2\{A;\phi\}^2 = 0 \tag{3.16}$$

where

$$\{A;\phi\} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left[\frac{A''(\phi)}{A'(\phi)}\right] - \frac{1}{2}\left[\frac{A''(\phi)}{A'(\phi)}\right]^2 \tag{3.17}$$

is the Schwarzian derivative (e.g. Weiss [2, 3], Hille [13]).

By regarding ϕ as a function of A, and using the result

$$\{A;\phi\} = -\frac{1}{[\phi'(A)]^2}\{\phi;A\} \tag{3.18}$$

equation (3.16) reduces to

$$-[\phi'(A)]^{-4}\frac{\mathrm{d}^2}{\mathrm{d}A^2}\{\phi;A\} = 0. \tag{3.19}$$

Equation (3.19) can be integrated twice with respect to A, to obtain

$$\{\phi;A\} = 2(k_1 A + k_2) \tag{3.20}$$

where k_1 and k_2 are arbitrary constants. Use of a standard transformation for Schwarzian differential equations (Hille [13]) shows that equation (3.20) has solutions of the form

$$\phi = Y_1/Y_2 \tag{3.21}$$

where Y_1 and Y_2 are two independent solutions of the linear second order differential equation

$$Y''(A) + (k_1 A + k_2) Y(A) = 0. \tag{3.22}$$

For $k_1 \neq 0$, equation (3.22) is a version of Airy's equation with general solution

$$Y = (\Delta A)^{1/2} \left\{ a J_{1/3} \left[\tfrac{2}{3} \sqrt{k_1} (\Delta A)^{3/2} \right] + b Y_{1/3} \left[\tfrac{2}{3} \sqrt{k_1} (\Delta A)^{3/2} \right] \right\} \tag{3.23}$$

where a and b are arbitrary constants

$$\Delta A = A + k_2/k_1 \tag{3.24}$$

and $J_{1/3}(x)$ and $Y_{1/3}(x)$ are Bessel functions of the first and second kind of order 1/3 (e.g. Abramowitz and Stegun [16]). For $k_1 = 0$ and $k_2 \neq 0$, equation (3.22) has solutions in terms of exponential functions, whereas for $k_1 = 0$ and $k_2 = 0$, the general solution is a linear function of A.

Use of equation (3.3) together with equations (3.13)–(3.15) gives the result (3.8) for $A(x, y, t)$. Finally, from equation (2.4) we obtain the solution

$$u_1 = -2 \frac{Y_2'(A)}{Y_2(A)} \frac{1}{y + g(t)} + \tfrac{1}{2} g''(t) y + m'(t) + g'(t) A - \left(\tfrac{1}{2} g'(t) + k_0 + A \right)^2 \tag{3.25}$$

of the 1+2D Burgers equation (1.1), where $Y_2(A)$ is a solution of equation (3.22). Use of equation (3.21) and the Painlevé expansion (2.8) yields a solution of the form (3.25) for u but with $Y_2(A)$ replaced by $Y_1(A)$, where $Y_1(A)$ and $Y_2(A)$ are independent solutions of equation (3.22).

3.4. Case $f_\phi = 0$

Proposition 2. Assume that $f = f(t)$. Then the 1+2D Burgers equation has solutions of the form

$$u = -2 \frac{\vartheta_h}{\vartheta} + f'(t) y - [f(t)]^2 \tag{3.26}$$

where

$$h = x - f(t) y \tag{3.27}$$

and ϑ satisfies the linear heat-like equation

$$\vartheta_t - \vartheta_{hh} + [p(t) h + q(t)] \vartheta = 0. \tag{3.28}$$

The functions $p(t)$ and $q(t)$ in the 'restricted' Bäcklund transformation (3.26)–(3.28) are arbitrary functions of t, whereas $f(t)$ is a differentiable function of t.

To derive the Bäcklund transformation (3.26)–(3.28), first notice from equation (3.3) that

$$X = f(t) y + h(\phi, t) \qquad X_\phi = h_\phi. \tag{3.29}$$

Hence equation (3.6) reduces to

$$(h_\phi)^3 h_{tt\phi} - 2h_\phi h_{\phi\phi\phi t} + 6h_{\phi\phi}h_{\phi\phi t} + 2h_{\phi\phi\phi}h_{\phi t} - 6\left(h_{\phi\phi}\right)^2 \frac{h_{\phi t}}{h_\phi}$$
$$+\left[\frac{\partial^2}{\partial\phi^2}\{h;\phi\} - 5\frac{h_{\phi\phi}}{h_\phi}\frac{\partial}{\partial\phi}\{h;\phi\} + 5\left(\frac{h_{\phi\phi}}{h_\phi}\right)^2\{h;\phi\} - 2\{h;\phi\}^2\right] = 0. \tag{3.30}$$

We observe that the 'steady-state' version of equation (3.30), obtained by setting time derivatives of h equal to zero, is exactly equation (3.16) for $A(\phi)$, except that $A(\phi)$ is now replaced by $h(\phi)$. This suggests converting equation (3.30) to an equation for $\phi(h,t)$, to obtain (see appendix 2)

$$\frac{\partial}{\partial h}\left\{\tilde{C}_t + \tilde{C}\tilde{C}_h - 2\tilde{C}_{hh} - \tilde{S}_h\right\} = 0 \tag{3.31}$$

where

$$\tilde{C} = -\phi_t/\phi_h \qquad \tilde{S} = \{\phi;h\} \tag{3.32}$$

are homographic invariants (compare with equations (1.8)).

Equation (3.31) is reminiscent of the one-dimensional Burgers equation. To make this connection more obvious, we write equation (3.31) in the form

$$\frac{\partial}{\partial h}\{\rho_t + \rho\rho_h - \rho_{hh}\} = 0 \tag{3.33}$$

where

$$\rho = \left(\phi_{hh} - \phi_t\right)/\phi_h. \tag{3.34}$$

If we set the term in the curly brackets of (3.33) equal to zero, then we obtain the one-dimensional Burgers equation for ρ. By means of the Cole–Hopf transformation

$$\rho = -2\frac{\psi_h}{\psi} \tag{3.35}$$

equation (3.33) reduces to

$$\frac{\partial^2}{\partial h^2}\left(\frac{\psi_t - \psi_{hh}}{\psi}\right) = 0. \tag{3.36}$$

Integration of (3.36) twice with respect to h yields a heat-like equation

$$\psi_t - \psi_{hh} + [p(t)h + q(t)]\psi = 0 \tag{3.37}$$

where $p(t)$ and $q(t)$ are arbitrary functions of t. In view of the definition of ρ above, equation (3.35) can be written as the heat-like equation for ϕ

$$\phi_t - \phi_{hh} - 2\frac{\psi_h}{\psi}\phi_h = 0. \tag{3.38}$$

Use of the truncated Painlevé expansion (2.8) and expression (2.4) for u_1 leads to solutions of the 2D Burgers equation with the form

$$u = -2\frac{\phi_h}{\phi} + u_1 \tag{3.39}$$

$$u_1 = -2\frac{\psi_h}{\psi} + f'(t)y - [f(t)]^2 \tag{3.40}$$

where h is defined in equation (3.27). On setting

$$\vartheta = \phi\psi \tag{3.41}$$

and using equations (3.37) and (3.38), the solution (3.39) for u can be expressed in the form of the Bäcklund transformation (3.26)–(3.28). The result (3.39) is an auto-Bäcklund transformation.

4. The invariant Painlevé approach

One of the more striking features of the Bäcklund transformation (3.26)–(3.28), was the role played by the homographic invariants $\tilde{C}$ and $\tilde{S}$ in the derivation (see equations (3.31) and (3.32)). This role becomes even more evident in the invariant Painlevé approach.

In the invariant Painlevé expansion introduced by Conte [5], the equivalent Painlevé expansions

$$u = \phi^{-p}\sum_{j=0}^{\infty} u_j\phi^j = \chi^{-p}\sum_{j=0}^{\infty}\tilde{u}_j\chi^j \tag{4.1}$$

are employed in which

$$\chi = \frac{\phi/\phi_x}{1 - [\phi_{xx}/(2\phi_x^2)]\,\phi} \tag{4.2}$$

is a new expansion parameter that preserves the singularity character of the singularity manifold with $\chi = 0$ when $\phi = 0$, and leads to Painlevé expansion coefficients $\tilde{u}_j$ that are homographic invariants. The invariance of the $\tilde{u}_j$ under the Möbius transformation (1.7) follows from noting that the gradient of χ is given by a Riccati equation in χ with coefficients that depend on the homographic invariants. In particular, if $\phi = \phi(x,y,t)$ and $\chi = \chi(x,y,t)$, then taking the gradient of χ using equation (4.2) gives

$$\begin{aligned}
\chi_t &= -C + C_x\chi - \tfrac{1}{2}(C_{xx} + CS)\chi^2 \\
\chi_x &= 1 + \tfrac{1}{2}S\chi^2 \\
\chi_y &= -W + W_x\chi - \tfrac{1}{2}(W_{xx} + WS)\chi^2
\end{aligned} \tag{4.3}$$

where C, W and S are the homographic invariants

$$C = -\phi_t/\phi_x \qquad W = -\phi_y/\phi_x \qquad S = \{\phi; x\}\,. \tag{4.4}$$

From the integrability conditions $\chi_{xt} = \chi_{tx}$, $\chi_{yx} = \chi_{xy}$ and $\chi_{yt} = \chi_{ty}$ for equations (4.3), we obtain the relations

$$\begin{aligned} &S_t + C_{3x} + 2C_x S + CS_x = 0 \\ &S_y + W_{3x} + 2W_x S + WS_x = 0 \\ &C_y - W_t - CW_x + C_x W = 0. \end{aligned} \tag{4.5}$$

Substitution of the invariant expansion (4.1) into the partial differential system of interest (namely equation (1.1) in our case), and balancing the powers of χ, taking into account the results (4.3), leads to recurrence relations for the $\tilde{u}_j$, and the Painlevé analysis proceeds in the usual manner. However, since the two expansions (4.1) are equivalent, there is a direct transformation between the u_j and $\tilde{u}_j$ coefficients obtained by equating powers of χ in equation (4.1). The form of the transformation between the u_j and $\tilde{u}_j$ depends on whether $j \leq p$ or $j > p$ (see Conte [5]). The transformations are

$$\begin{aligned} \tilde{u}_j &= \sum_{k=0}^{j} \frac{(-k+p)!}{(-j+p)!(j-k)!} \phi_x^{k-p} \left(\frac{\phi_{xx}}{2\phi_x}\right)^{j-k} u_k \qquad \text{if} \quad j \leq p \\ \tilde{u}_j &= \sum_{k=1+p}^{j} \frac{(-1)^{j-k}(j-p-1)!}{(k-p-1)!(j-k)!} \phi_x^{k-p} \left(\frac{\phi_{xx}}{2\phi_x}\right)^{j-k} u_k \qquad \text{if} \quad j > p. \end{aligned} \tag{4.6}$$

By applying the results (4.6) to the truncated 1+2D Burgers Painlevé expansion of section 2, with $p = 1$, we obtain

$$u_0 = -2\phi_x \qquad u_1 = (\phi_{xx}/\phi_x) - (\phi_y/\phi_x)^2 - (\phi_t/\phi_x) \tag{4.7}$$

and hence

$$\tilde{u}_0 = u_0/\phi_x = -2 \tag{4.8}$$

$$\tilde{u}_1 = u_0\phi_{xx}/(2\phi_x^2) + u_1 = -(\phi_y/\phi_x)^2 - (\phi_t/\phi_x) \tag{4.9}$$

for the truncated invariant expansion coefficients. Note in particular that $\tilde{u}_0$ is constant and

$$\tilde{u}_1 = C - W^2 \tag{4.10}$$

is invariant under the Möbius transformation (1.7). The truncated invariant expansion is

$$u = \tilde{u}_0/\chi + \tilde{u}_1 = -2/\chi + C - W^2. \tag{4.11}$$

Similarly, from (4.9) and (4.10),

$$u_1 = C - W^2 + \phi_{xx}/\phi_x. \tag{4.12}$$

As in the non-invariant analysis, there are two constraints placed on ϕ. The first is the compatability condition (2.5) at the $j = 2$ resonance, which can be written as

$$W_y + WW_x = 0. \tag{4.13}$$

The second constraint can be obtained either by substituting the expression (4.12) for u_1 in equation (2.7) or by requiring that the expansion (4.11) for u satisfy the 1+2D Burgers equation (1.1). We obtain

$$\begin{aligned}\frac{\partial}{\partial x}\big(-2WW_t + 3WW_{xx} + \tfrac{7}{2}W_x^2 + C_t + CC_x - 2C_{xx}\\ - 2WW_xC - W^2C_x - W^2S - S_x\big) - WS_y + C_{yy} = 0.\end{aligned} \tag{4.14}$$

Use of the results (4.5) and (4.13) to eliminate the y derivatives in equation (4.14) gives

$$\begin{aligned}\frac{\partial}{\partial x}\left(-2WW_t + 4WW_{xx} + 3W_x^2 + C_t + CC_x - 2C_{xx} - S_x\right)\\ - \left(4WW_xC_x + 2CW_x^2 + 4CWW_{xx} + 2WW_{xt}\right) = 0.\end{aligned} \tag{4.15}$$

Searching for solutions of equations (4.13) and (4.15) with $W_x = 0$, equation (4.13) implies $W_y = 0$, and hence $W = f(t)$. Equation (4.15) then reduces to

$$\frac{\partial}{\partial x}(C_t + CC_x - 2C_{xx} - S_x) = 0. \tag{4.16}$$

This equation is a disguised form of equation (3.31), which leads directly to the Bäcklund transformation (3.26)–(3.28). By integrating the equation

$$W = f(t) = -\phi_y/\phi_x \tag{4.17}$$

for ϕ, we obtain

$$\phi = \phi(h,t) \qquad h = x - f(t)y. \tag{4.18}$$

On using h and t as independent variables, equation (4.16) reduces to equation (3.31). This is clearly a much more elegant and direct path to the Bäcklund transformation than that developed in section 3, but the choice of the ansatz $W = f(t)$, was in fact suggested by the more tortuous analysis of that section.

Similarly, searching for solutions of equation (4.15) of the form $\phi = \Phi(A)$ with $A_{xx} = 0$, leads to the solution (3.7) of the 1+2D Burgers equation. This solution is also characterized by the conditions

$$W_{xx} = S_{xx} = C_{xx} = 0. \tag{4.19}$$

However, it is not evident *a priori* that the above solution ansatz would be successful.

5. Concluding remarks

In this paper, we have carried out a Painlevé analysis for the two-dimensional Burgers equation. The function ϕ, defining the singularity manifold $\phi(x,y,t) = 0$, was found to be subject to two constraints, namely the quasi-linear partial differential equation (2.5) which determined the functional form of ϕ, and the further constraint (2.7) that the second coefficient $u_1(x,y,t)$ of the Painlevé expansion itself satisfy the 2D

Burgers equation. The singularity analysis showed that a subclass of the solutions of (1.1) may be mapped onto a linear heat-like equation by means of the Bäcklund transformation (1.2)–(1.4). The singularity analysis also yielded a further class of solutions, obtained by solving a Schwarzian differential equation (equations (3.7)–(3.9)). A clearer derivation of the Bäcklund transformation was provided (section 4) by using the invariant Painlevé analysis developed by Conte [5], in which the coefficients of the expansion are homographic invariants. The main advantage of this approach is that it uncovers the homographic invariant sub-structure in the analysis. However, this approach does not by itself lead to a systematic uncovering of all possible solutions for the truncated expansion.

It is clear that the Bäcklund transformation can be used to construct multiple shock solutions of the 2D Burgers equation (compare for example the construction of multiple shock solutions of the 1D Burgers equation given by Whitham [17], ch. 4). Also of interest would be a clarification of the relationship of the present work with the group theoretical and Lie algebraic properties of the 2D Burgers equation (see, e.g., Harrison and Estabrook [18], Wahlquist and Estabrook [19]).

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Appendix 1.

In this appendix, we list the set of nine determining equations for f and g obtained by equating powers of $X_\phi = f_\phi(y+g)$ to zero in equation (3.6). This procedure leads to the solutions (3.13)–(3.15) for f, g and h. The equations are

$$f_{tt\phi} = 0 \tag{A1.1}$$

$$\frac{\partial}{\partial t}\left[(f_\phi)^2(g_t - 2f)\right] = 0 \tag{A1.2}$$

$$2\frac{\partial}{\partial t}\{f;\phi\} = 0 \tag{A1.3}$$

$$2f_\phi\frac{\partial}{\partial t}\left[\frac{f_{\phi\phi}g_\phi - f_\phi g_{\phi\phi}}{f_\phi}\right] = 0 \tag{A1.4}$$

$$f_\phi^3\left\{2\frac{\partial}{\partial\phi}\left[\frac{g_\phi}{f_\phi}(g_t - 2f)\right] + 4\frac{g_\phi}{f_\phi}\frac{\partial}{\partial\phi}(g_t - 2f) + 2g_\phi\right\} + \frac{\partial^2}{\partial\phi^2}\{f;\phi\}$$
$$- 5\frac{f_{\phi\phi}}{f_\phi}\frac{\partial}{\partial\phi}\{f;\phi\} + 5\left(\frac{f_{\phi\phi}}{f_\phi}\right)^2\{f;\phi\} - 2\{f;\phi\}^2 = 0 \tag{A1.5}$$

$$6g_\phi f_\phi \left[-(f_{4\phi}/f_\phi) + 7(f_{\phi\phi}f_{3\phi}/f_\phi^2) - 8\left(f_{\phi\phi}/f_\phi\right)^3\right] - 6g_{\phi\phi}f_\phi\left[(f_{3\phi}/f_\phi) - 3\left(f_{\phi\phi}/f_\phi\right)^2\right]$$
$$- 6g_{3\phi}f_{\phi\phi} + g_{4\phi}f_\phi - 6g_\phi^2 f_\phi^3(g_t - 2f) = 0 \qquad (A1.6)$$

$$6g_\phi^2 f_\phi^2\left[3(f_{3\phi}/f_\phi) - 8\left(f_{\phi\phi}/f_\phi\right)^2\right] + 42g_\phi g_{\phi\phi}f_\phi f_{\phi\phi} - 6g_{\phi\phi}^2 f_\phi^2 - 10g_\phi g_{3\phi}f_\phi^2 = 0 \qquad (A1.7)$$

$$48f_\phi^2 g_\phi^2[f_\phi g_{\phi\phi} - g_\phi f_{\phi\phi}] = 0 \qquad (A1.8)$$

$$- 36(g_\phi f_\phi)^4 = 0. \qquad (A1.9)$$

A major constraint on the solutions of equations (A1.1)–(A1.9) is obtained from equation (A1.9), which requires that $g_\phi = 0$ since $f_\phi \neq 0$ was assumed from the outset. The solving of equations (A1.1)–(A1.8) with $g_\phi = 0$ yields the solutions (3.13)–(3.15).

Appendix 2.

By considering ϕ as a function of h and t, equation (3.30) can be written as

$$-\frac{\partial}{\partial h}\left[\frac{\phi_h^2\phi_{tt} + \phi_t^2\phi_{hh} - 2\phi_t\phi_h\phi_{ht}}{\phi_h^3}\right] + 2\frac{\phi_{3ht}}{\phi_h} - 6\frac{\phi_{ht}\phi_{3h}}{\phi_h^2} - 6\frac{\phi_{hh}\phi_{hht}}{\phi_h^2} + 12\frac{\phi_{ht}\phi_{hh}^2}{\phi_h^3} - 2\frac{\phi_t\phi_{4h}}{\phi_h^2}$$
$$+ 12\frac{\phi_t\phi_{hh}\phi_{3h}}{\phi_h^3} - 12\frac{\phi_t\phi_{hh}^3}{\phi_h^4} - \frac{\partial^2}{\partial h^2}\{\phi; h\} = 0. \qquad (A2.1)$$

Manipulation of equation (A2.1) leads to equation (3.31).

References

[1] Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
[2] Weiss J 1984 *J. Math. Phys.* **25** 2226
[3] Weiss J 1984 *J. Math. Phys.* **25** 13
[4] Newell A C, Tabor M and Zeng Y B 1987 *Physica 29D* 1–68
[5] Conte R 1989 *Phys. Lett.* **140A** 383
[6] Bartucelli M, Pantano P and Brugarino T 1983 *Lett. Nuovo Cimento* **37** 433–8
[7] Zabolotskaya E A and Khoklov R V 1969 *Sov. Phys. Acoustics* **8** 35
[8] Rudenko O V and Soluyan S I 1977 *Theoretical Foundations of Nonlinear Acoustics (Studies in Soviet Science)* (New York: Plenum)
[9] Crighton D G 1979 *Ann. Rev. Fluid Mech.* **11** 11
[10] Hunter J K 1988 *SIAM J. Appl. Math.* **48** 1
[11] Zank G P and Webb G M 1990 *J. Plasma Phys.* in press
[12] Segur H 1986 *Physica* **18D** 1–12
[13] Hille E 1969 *Lectures on Ordinary Differential Equations* (Reading, MA: Addison Wesley)
[14] Cariello F and Tabor M 1989 *Physica* **39D** 77–94
[15] Sneddon I N 1957 *Elements of Partial Differential Equations* (New York: McGraw Hill)
[16] Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
[17] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
[18] Harrison B K and Estabrook F B 1971 *J. Math. Phys.* **12** 653
[19] Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1